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VISCOUS THEORY OF LIFT ON BODIES OF REVOLUTION

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The viscous theory of unsteady lift on bodies of revolution is developed. The exact potential theory is derived and reduced to an integral equation for an unknown load function in terms of a generalized upwash function. The integral theory is formally the same as two-dimensional wing theory and the viscous modification of the kernel function is based on the principles of wing theory and detailed analysis of a ring-wing. Numerical results are presented for the ring-wing in steady flow that are in

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SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered) agreement with the predictions of slender body theory for small diameter to length ratios. Asymptotic results for very large diameter to length ratios are in exact agreement with previously reported theoretical results and approximate agreement with experiment.

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NOMENCLATURE

a	radius of ring-wing
C_M	moment coefficient of ing-wing, see Equation (3.12)
C_N	normal force coefficient of ring-wing, see Equation (3.11)
$E(k)$	see Equation (2.29)
$\mathscr{F}$	see Equation (2.57)
G	Green's function, see Equations (2.16) and (2.17)
$\mathscr{G}$	see Equation (2.26)
$H(x)$	heaviside step function
$\vec{i},\vec{j},\vec{k}$	see Figure 2.1
k	see Equations (2.44) and (3.3)
k'	see Equation (3.10)
$K(k)$	see Equation (2.30)
K_0	modified Bessel function
$\mathscr{K}_1(X)$	integrated kernel function, see Equation (3.8)
$\mathscr{K}_v$	viscous kernel functions, see Equations (2.47) and (2.48)
$\mathscr{K}(X,\xi)$ or $\mathscr{K}(x)$	kernel functions
ℓ	length of ring-wing, see Figure 2.2
$\ell(\xi,\theta)$	see Equations (2.14) and (2.24)
L	reference length
$\mathscr{L}$	see Equation (2.24)
$\vec{n}$	$= \vec{N}/\lvert\vec{N}\rvert$ unit normal on S
$\vec{N}$	normal to control surface S, see Figure 2.1
p	perturbation pressure
q_0	$= \lvert\vec{v}_0\rvert$
$\hat{r},\hat{\theta}$	unit vectors, see inset in Figure 2.1
$R(x)$	body or control surface radius
S	see Figure 2.1
T_n	Chebyshev polynomial of the first kind of degree n, see Reference 6
$T(k')$	see Equation (3.9)
$T(x,\xi)$	see Equation (2.15)
V_∞	free stream velocity
$\vec{v}_0$	velocity field of non-lifting axial flow
$\mathscr{W}_g$	generalized upwash, see Equation (2.36)
$\mathscr{W}_g^0, \mathscr{W}_g^1$	components of $\mathscr{W}_g$, see Equations (2.38), (2.39) and (2.40)

$\mathcal{W}(x)$	upwash function, see Equation (2.2)
x,r,θ	cylindrical coordinates, see Figure 2.1
$\vec{x}$	$= (x,y,z)$ Cartesian coordinates, see Figure 2.1
X	$2x/\ell$
X_N, X_T	axial coordinates of body fore and aft ends, see Figure 2.1
Re	Reynolds number, see Equation (2.49)
∇^2	Laplace operator
$\alpha(x)$	local angle attack of ring-wing axis, see Equation (3.6)
δ^*	boundary layer displacement thickness
$\delta(\vec{x})$	Dirac delta function
ν	kinematic coefficient of viscosity
ρ_∞	free stream density
σ	Re/4
τ	$4a/\ell$
ϕ	velocity potential
Φ	see Equation (2.22)
ω	frequency of simple harmonic oscillation

I. INTRODUCTION

In Reference 1 the viscous theory of unsteady lift on two-dimensional airfoils of arbitrary thickness distribution is developed in detail. Extensive numerical examples are given for steady and unsteady flow that compare favorably with experimental results. Both the d'Alembert paradox and the fatness paradox (see Reference 2) are resolved, and most important, the role of viscosity in establishing a unique value of the circulatory part of the lift is completely understood within the framework of this theory. The concept of viscous thin airfoil theory is directly applicable to other lifting bodies, e.g., three-dimensional wings and bodies of revolution. In a recent report (Reference 3), completed under contract to NASA, the preliminary application of the viscous theory to a rectangular wing is presented. The numerical results are encouraging and reinforce the basic principles of the two-dimensional theory.

The present report is directed towards the application of the viscous theory to bodies of revolution. The complete unsteady potential theory is presented and subsequently corrected for the effect of viscosity. The theory is applied to the simple case of a ring-wing. Both numerical results and asymptotic results are presented. For very slender ring-wings we recover the well known results of slender body theory. For very large diameter to length ratios we obtain a simple asymptotic result that is in exact agreement with a theoretical result of Ribner (Reference 4). The theoretical result is also in approximate agreement with experimental results of Flatau (Reference 5).

II. VISCOUS THEORY OF LIFT FOR BODIES OF REVOLUTION IN INCOMPRESSIBLE FLOW

A. Preliminary Remarks

Consider a body of revolution in a high Reynolds number ($Re > 10^6$) incompressible flow (see Figure 2.1). In the absence of any disturbance that would produce a transverse force (lift), the body axis is aligned with the free stream and the mean steady flow is characterized by the velocity field $\vec{v}_0$ that we assume to be given. Suppose that the body with its boundary layer and wake are enclosed by a cylindrical control surface S. In the domain D exterior to S the flow is free of vorticity and thus has a velocity potential. Forward of the plane P in Figure 2.1, the distance between the surface S and the actual body surface will be of the order of the boundary layer thickness. The plane P can be imagined to lie a few percent of the body length forward of the aft end. Immediately downstream of the body, the surface S is faired smoothly into a cylinder of constant radius that bounds the wake. As the Reynolds number is increased beyond all bound the radius of the wake cylinder will tend to a constant.

Forward of the plane P, the normal momentum and pressure at a point on S will differ from the corresponding values on the actual body surface by terms that are of the order δ^*/L where δ^* is the turbulent boundary layer displacement thickness and L is the local radius of curvature or the body length, whichever is smaller. Terms of this order will be neglected in the present theory. Downstream of the body the transverse load on S must tend to zero.

To derive the viscous theory, the first step is to formulate the potential flow problem in D and then recast it in the form of an integral equation for the transverse load function (to be defined), given that the normal momentum (upwash) on S is specified. The local singularity in the load-upwash kernel function is then modified (reduced in strength) to account for the direct effect of viscosity. The principle for introducing the local effect of viscosity is based on our previous work with viscous wing theory (Reference 1).

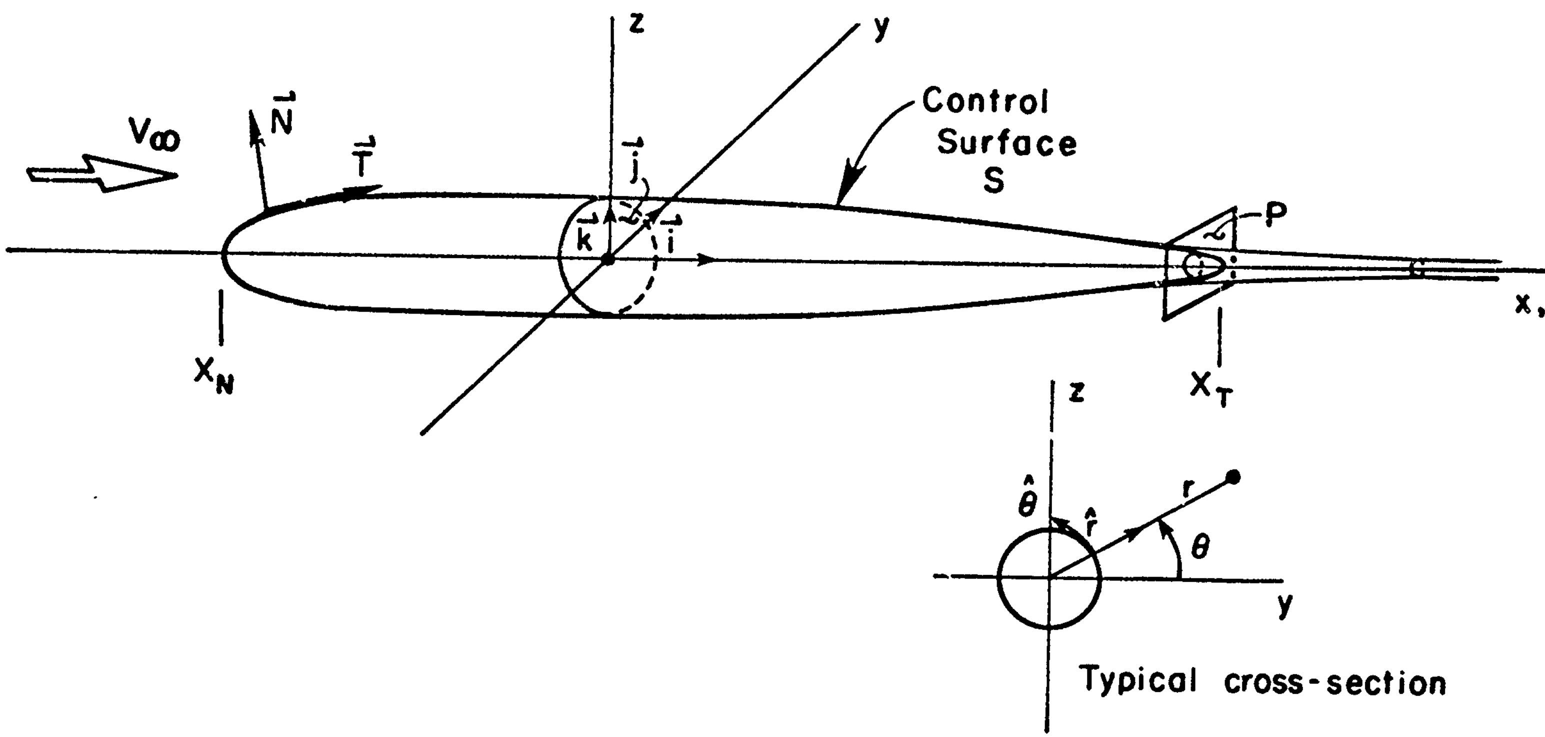

Figure 2.1 Body of Revolution and Enveloping Control Surface

B. Potential Flow Problem

Let ϕ denote the perturbation potential due to an unsteady simple harmonic transverse momentum disturbance on S. The boundary value problem for ϕ is the following:

$$\nabla^2\phi = 0 \quad \text{in } D \tag{2.1}$$

$$\frac{\partial\phi}{\partial N} = \mathcal{W}(x)\sin\theta \quad \text{on } S: \ r = R(x) \tag{2.2}$$

$$\text{grad } \phi \sim 0 \ \text{ at } \infty \quad \text{(except near the wake)} \tag{2.3}$$

and for the perturbation pressure

$$p = -\rho_\infty \frac{D_o\phi}{Dx} \tag{2.4}$$

where

$$\frac{D_o}{Dx} = \vec{v}_o \cdot \frac{\partial}{\partial\vec{x}} + i\omega \tag{2.5}$$

and ω is the radian frequency of the specified transverse momentum disturbance. Also note that

$$\frac{\partial}{\partial N} = \vec{N} \cdot \frac{\partial}{\partial\vec{x}} = -R'\frac{\partial}{\partial x} + \frac{\partial}{\partial r} \tag{2.6}$$

where

$$\vec{N} = - R'(x)\hat{r} + \vec{i} \tag{2.7}$$

is normal to the surface S. The unit normal on S is given by

$$\vec{n} = \frac{\vec{N}}{|\vec{N}|} = \frac{\vec{N}}{\sqrt{1 + R'^2}} \tag{2.8}$$

We use Cartesian and cylindrical coordinates interchangeably in the subsequent development.

Before we proceed with the representation of ϕ, we first consider the inverse of the relation (2.4). Note that

$$\vec{v}_0 \cdot \frac{\partial}{\partial \vec{x}} = q_0 \frac{d}{ds} \tag{2.9}$$

where

$$ds = s'dx = \sqrt{1 + R'^2}\ dx \tag{2.10}$$

and

$$q_0 = |\vec{v}_0| \tag{2.11}$$

the speed of the nonlifting flow. Thus, we can write

$$\frac{d\phi}{ds} + \frac{i\omega}{q_0} \phi = - \frac{p}{\rho_\infty q_0} \tag{2.12}$$

The solution of this equation for ϕ is straightforward. The final result is

$$\phi(x,\theta) = v_\infty \int_{X_N}^{x} \ell(\xi,\theta)e^{-i\omega T(x,\xi)}\, d\xi \qquad (2.13)$$

where

$$\ell = -\frac{p\sqrt{1+R'^2}}{\rho_\infty q_0 v_\infty} \qquad (2.14)$$

and

$$T(x,\xi) = \int_{\xi}^{x} \frac{\sqrt{1+R'^2}}{q_0}\, d\xi \qquad (2.15)$$

We refer to ℓ as the generalized load function since it defines completely the local lift force on the surface S. The two point function $T(x,\xi)$ is the time required for a vortical disturbance produced by a harmonic load at station ξ to convect along the surface at speed q_0 to station x. It is the most important unsteady effect of body shape on the aerodynamic loads.

C. The Load-Upwash Integral Equation

Consider the Green's function

$$G = \frac{1}{|\vec{x}-\vec{\xi}|} \qquad (2.16)$$

that satisfies the Poisson equation

$$\nabla^2 G = -4\pi\, \delta(\vec{x}-\vec{\xi}) \qquad (2.17)$$

where $\delta(\vec{x})$ is the Dirac delta function. Form the identity

$$G\nabla^2\phi - \phi\nabla^2 G = 4\pi\, \phi\, \delta(\vec{x}-\vec{\xi}) \qquad (2.18)$$

and integrate with respect to coordinates $\vec{\xi}$ over the domain D. After the application of the divergence theorem, the final result can be expressed in the following form:

$$\phi(\vec{x}) = \frac{1}{4\pi} \oiint_S \left(\phi \frac{\partial G}{\partial N'} - G \frac{\partial \phi}{\partial N'}\right) dA \tag{2.19}$$

where

$$dA = R(\xi)d\xi\, d\theta' \tag{2.20}$$

and

$$\frac{\partial}{\partial N'} = - R'(\xi) \frac{\partial}{\partial \xi} + \frac{\partial}{\partial \rho} \tag{2.21}$$

The next step is to eliminate the polar angle θ from the problem. It follows from the boundary condition (2.2) that all dependent variables are proportional to sin θ. Thus, we write

$$\phi = \Phi(x,r)\sin\theta \tag{2.22}$$

$$p = P(x,r)\sin\theta \tag{2.23}$$

and also for the load function

$$\ell = \mathscr{L}(x,r)\sin\theta \tag{2.24}$$

Substitute Equations (2.22) and (2.2) into (2.19) to obtain

$$\Phi(x,r) = \frac{1}{4\pi} \int_{X_N}^{\infty} R(\xi)d\xi \left(\Phi \frac{\partial \mathscr{G}}{\partial N'} - \mathscr{G} \cdot \mathscr{W}\right) \tag{2.25}$$

where

$$\mathscr{G} = \int_0^{2\pi} \frac{\cos\theta \, d\theta}{\sqrt{x^2+r^2+\rho^2-2r\rho\cos\theta}} \tag{2.26}$$

$$= -\frac{4}{\sqrt{x^2+(r+\rho)^2}} \left[\left(1 - \frac{2}{k^2}\right) K(k) + \frac{2}{k^2} E(k)\right] \tag{2.27}$$

with

$$k^2 = \frac{4r\rho}{x^2+(r+\rho)^2} \tag{2.28}$$

$$E(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\theta} \, d\theta \tag{2.29}$$

$$K(k) = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1-k^2\sin^2\theta}} \tag{2.30}$$

The functions E(k) and K(k) are standard elliptic integrals (see Reference 6). Now replace Φ in the first term of (2.25) with the integral over the load function; i.e.,

$$\Phi(\xi,R(\xi)) = v_\infty \int_{X_N}^{\xi} \mathscr{L}(\xi')e^{-i\omega T(\xi,\xi')}\, d\xi' \tag{2.31}$$

The result is

$$\Phi(x,r) = \frac{v_\infty}{4\pi} \int_{X_N}^{\infty} \mathscr{L}(\xi)d\xi \int_{\xi}^{\infty} e^{-i\omega T(\xi',\xi)} \frac{\partial \mathscr{G}}{\partial N'} R(\xi')d\xi' \tag{2.32}$$

$$- \frac{1}{4\pi} \int_{X_N}^{\infty} \mathscr{G}\mathscr{W}R(\xi)d\xi$$

Finally, we compute the normal derivative of the last result on the surface S. The last term has a Cauchy principle value so that

$$\lim_{r\to R(x)} \frac{1}{4\pi} \int_{X_N}^{\infty} \frac{\partial \mathscr{G}}{\partial N} \mathscr{W}Rd\xi = -\frac{1}{2}\mathscr{W} + \frac{1}{4\pi} ⨍_{X_N}^{\infty} \frac{\partial \mathscr{G}}{\partial N} \mathscr{W}Rd\xi \tag{2.33}$$

and the slash through the integral indicates the usual limit operation; i.e.,

$$⨍_{X_N}^{\infty} d\xi = \lim_{\varepsilon\to 0} \int_{X_N}^{X-\varepsilon} \int_{X+\varepsilon}^{\infty} d\xi \tag{2.34}$$

The integral equation for $\mathscr{L}$ can thus be written in the following form:

$$\frac{1}{\pi} ⨍_{X_N}^{X_T} \mathscr{L}(\xi)\mathscr{K}(x,\xi)d\xi = \mathscr{W}_g(x)/v_\infty \tag{2.35}$$

where

$$\mathscr{W}_g(x) = \mathscr{W}(x) + \frac{1}{2\pi} \int_{X_N}^{\infty} \frac{\partial \mathscr{G}}{\partial N} \cdot \mathscr{W} R d\xi \tag{2.36}$$

and

$$\mathscr{K}(x,\xi) = \lim_{r \to R(x)} \int_{\xi}^{\infty} e^{-i\omega T(\xi',\xi)} \frac{\partial^2 \mathscr{G}}{\partial N \partial N'} R(\xi') d\xi' \tag{2.37}$$

The upper limit of integration in Equation (2.35) is X_T since the transverse load must vanish on the wake cylinder (to the order we are working). The generalized upwash $\mathscr{W}_g(x)$ can be separated into two parts; i.e.,

$$\mathscr{W}_g(x) = \mathscr{W}_g^0(x) + \mathscr{W}_g^1(x) \tag{2.38}$$

where

$$\mathscr{W}_g^0(x) = \mathscr{W}(x) + \frac{1}{2\pi} \int_{X_N}^{X_T} \frac{\partial \mathscr{G}}{\partial N} \cdot \mathscr{W} R d\xi \tag{2.39}$$

and

$$\mathscr{W}_g^1(x) = \frac{1}{2\pi} \int_{X_T}^{\infty} \frac{\partial \mathscr{G}}{\partial N} \cdot \mathscr{W} R d\xi \tag{2.40}$$

The first part, $\mathscr{W}_g^0$, is completely determined in terms of the known upwash on the surface of the body. The second part, $\mathscr{W}_g^1$, can only be determined after $\mathscr{W}$ is calculated on the wake cylinder. The actual numerical process must be carried out iteratively. First assume $\mathscr{W}_g^1(x)$ to be zero and solve Equation (2.35) for the load function. Then calculate $\mathscr{W}_g$ on the wake cylinder by using

Equation (2.35) with $x > X_T$. Then calculate $\mathscr{W}_g^1$ with Equation (2.40) and correct the generalized upwash with Equation (2.38). If the wake correction is shown to be small, the process can be terminated. Otherwise, it will be necessary to recalculate a new load function.

We emphasize that the foregoing problem is not well posed as yet. The reason is that the kernel function (2.37) has been derived with potential theory and has a Cauchy singularity (see below Equation (2.46)). The integral equation (2.35) has an eigensolution. The purpose of the viscous theory is to correct the singularity in the kernel such that Equation (2.35) has a unique solution. We turn now to an explicit evaluation of the kernel for the case of a ring-wing. The reason is to illustrate the basic structure of the kernel and, in particular, the singularity. Also we show how conventional slender body theory "tampers" with the singularity in such a way that a unique solution is obtained in terms of an effective base area.

D. Properties of the Kernel Function for a Ring-Wing in Steady Flow

Consider the ring-wing shown in Figure 2.2. For $\omega = 0$ the kernel (2.37) can be expressed in the following form:

$$\mathscr{K}(x,\xi) = a\mathscr{K}(x-\xi) \tag{2.41}$$

where

$$\mathscr{K}(x) = \lim_{r\to\rho\to a} \int_{-\infty}^{x} \frac{\partial^2 \mathscr{G}}{\partial r \partial \rho} d\xi \tag{2.42}$$

The detailed evaluation of the ring-wing kernel is given in the Appendix. The final result is

$$\mathscr{K}(x) = -\frac{1}{a^2}\left\{\pi + \left[\frac{E(k)}{\sqrt{1-k^2}}\left(\frac{z}{k} - k\right)^2 - \frac{4}{k^2}\ (1-k^2)^{3/2}\ K(k)\right] \mathrm{sgn}\ x\right\} \tag{2.43}$$

RING-WING

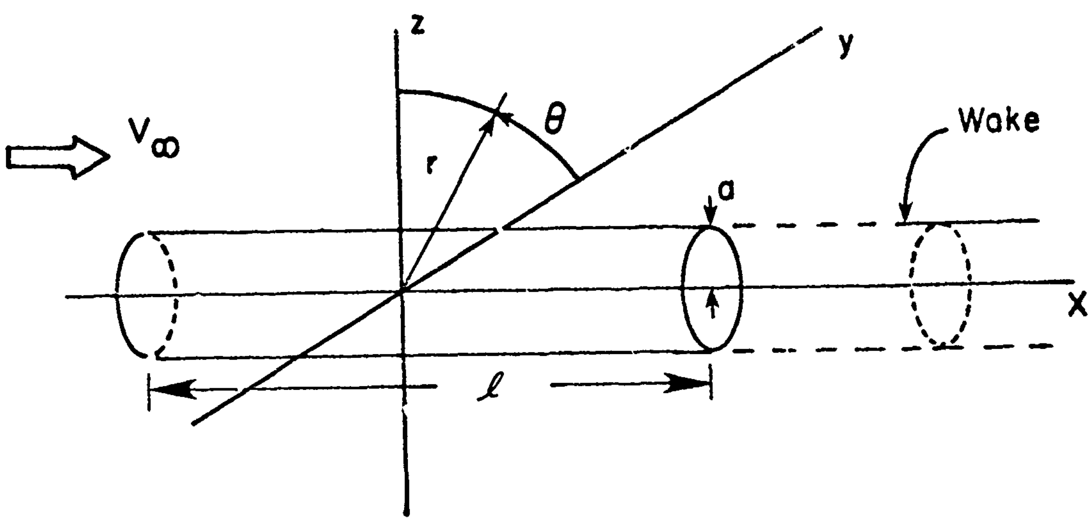

Figure 2.2 The Ring-Wing in Steady Flow

where

$$k = \frac{2a}{\sqrt{x^2+4a^2}} \tag{2.44}$$

and E(k), K(k) are the elliptic integrals given by Equations (2.29) and (2.30). Also we have the following two asymptotic results for large and small x, respectively:

<u>For x > 2a</u>

$$\mathscr{K}(x) \cong -\frac{2\pi}{a^2} H(x) \tag{2.45}$$

<u>For x << 2a</u>

$$\mathscr{K}(x) \cong -\frac{2}{a} \cdot \frac{1}{x} \tag{2.46}$$

Numerical calculations of the load distribution on a ring-wing may be found in Section III. Here we want to synthesize the viscous correction to the kernel that is necessary to obtain a unique solution of the integral equation. Also we can compare the viscous and inviscid kernels with the asymptotic kernel (2.45) that yields the results of slender body theory.

E. The Viscous Slender Body Kernel Function

The asymptotic behavior of the kernel function for small x (i.e., x << 2a, see Equation (2.46)) dictates a fundamental mathematical property of the integral equation (2.35). That is, the equation has an eigensolution whose magnitude is proportional to the lift on the body. The problem is completely analogous to the two-dimensional wing theory discussed in Reference 1. There we have shown rigorously that the correct viscous modification of the kernel is to convert the Cauchy singularity into a weaker logarithmic singularity on the Stokes scale. For the ring-wing, the correct modification of the asymptotic result (2.46) is

$$\mathscr{K}_v(x) \cong -\frac{2}{a}\frac{\partial}{\partial x}\left(\ln|x| + e^{v_\infty x/2\nu}\ K_0(v_\infty|x|/2\nu)\right) \tag{2.47}$$

where K_0 is the modified Bessel function. For $x << 2a$ the viscous ring-wing kernel behaves like

$$\mathscr{K}_v \cong \frac{2}{a}\ln|x| \ , \quad x << 2a \tag{2.48}$$

The argument for correcting the ring-wing kernel for viscosity is based entirely on our previous work on the two-dimensional wing. So long as the effect of viscosity is confined to a circumferential region that is small compared to the smallest geometric dimensions, the ring-wing behaves locally like a two-dimensional wing. If we introduce the following dimensionless parameters,

$$Re = \frac{v_\infty \ell}{\nu} \tag{2.49}$$

and

$$\tau = \frac{4a}{\ell} \tag{2.50}$$

then a necessary requirement for the validity of Equation (2.47) is that

$$\frac{1}{\sqrt{Re}} << \tau \tag{2.51}$$

This result says that the Stokes viscous length scale must be much smaller than the smallest geometric length scale.

Another remarkable property of the kernel function is that the singularity near the load is not changed when the flow becomes unsteady. Also the complete unsteady potential theory kernel for an arbitrary body of revolution (see Equation (2.37)) has a local Cauchy singularity like a ring-wing of radius R(x). Thus, we can write down the complete viscous slender body kernel without further ado:

$$\mathcal{K}_v(x,\xi) = \lim_{r\to R(x)} \int_{\xi}^{\infty} e^{-i\omega T(\xi',\xi)} \frac{\partial^2 \mathcal{G}}{\partial N \partial N'} R(\xi')d\xi' \tag{2.52}$$

$$- \frac{2}{R(x)} e^{V_\infty(x-\xi)/2\nu} K_0(V_\infty|x-\xi|/2\nu)$$

The complete problem of calculating the lift (steady or unsteady) on a body of revolution is to solve the integral equation (2.35) with the viscous kernel (2.52). The solution is unique because the kernel has the correct logarithmic singularity. Specific calculations for a ring-wing are given in Section III.

F. The Kernel of Slender Body Theory

We assume that the asymptotic result (2.45) is valid for all x and furthermore, extend it to a slender body of revolution by replacing a with R(x). Then the integral equation (2.35) can be written in the form

$$\frac{1}{2\pi} \int_{X_N}^{x} \left(\frac{\mathcal{F}(\xi)}{\pi}\right) \cdot \left(- \frac{2\pi}{R^2(x)} H(x-\xi)\right) d\xi = \mathcal{W}_g(x) \tag{2.53}$$

where

$$\mathcal{F}(\xi) = \pi R(\xi)\mathcal{L}(\xi) \tag{2.54}$$

is local the force per unit length on the body. The solution of Equation (2.53) is

$$\mathscr{F}(x) = -\frac{d}{dx}\left[\pi R^2(x)\mathscr{W}_g(x)\right] \tag{2.55}$$

For a body at constant angle of attack, we have

$$\mathscr{W}_g = -\alpha \tag{2.56}$$

and

$$\mathscr{F}(x) = \alpha\frac{dS}{dx} \tag{2.57}$$

where

$$S = \pi R^2 \tag{2.58}$$

is the local cross-sectional area of the body. We recognize the last result as the familiar formula of classical slender body theory; i.e., the local lift force on a body of revolution at constant angle of attack is proportional to the derivative of the cross-sectional area. The slender body formula yields a reasonably good approximation of the sectional lift on the forward portion of the body. Unfortunately, for closed bodies of revolution, it leads to the embarrassing result that the total lift is zero. In application of the theory it has become cumstomary to introduce an "effective base area" and to correlate measured lift data in terms of this artificial parameter. With the foregoing general theory we can offer a plausible explanation of this apparent uniqueness that is brought about by introducing the familiar slender body approximation into the potential flow theory. We use the term apparent uniqueness because the unknown lift is simply traded off for an equally unknown base area.

The potential flow kernel, the viscous kernel and the slender body approximation thereof are plotted in Figure 2.3. From the point of view of the theory of integral equations, the slender body approximation represents a rather crude elimination of the fundamental Cauchy singularity in the exact potential flow theory. Indeed, this step eliminates the possibility of an eigensolution and thus yields a unique value for the local normal force. The approximation is not too bad but does not lead to a unique value for the total lift. The viscous modification of the Cauchy singularity is the correct way to proceed. A unique value of the total lift can be calculated without the device of an artificial base area.

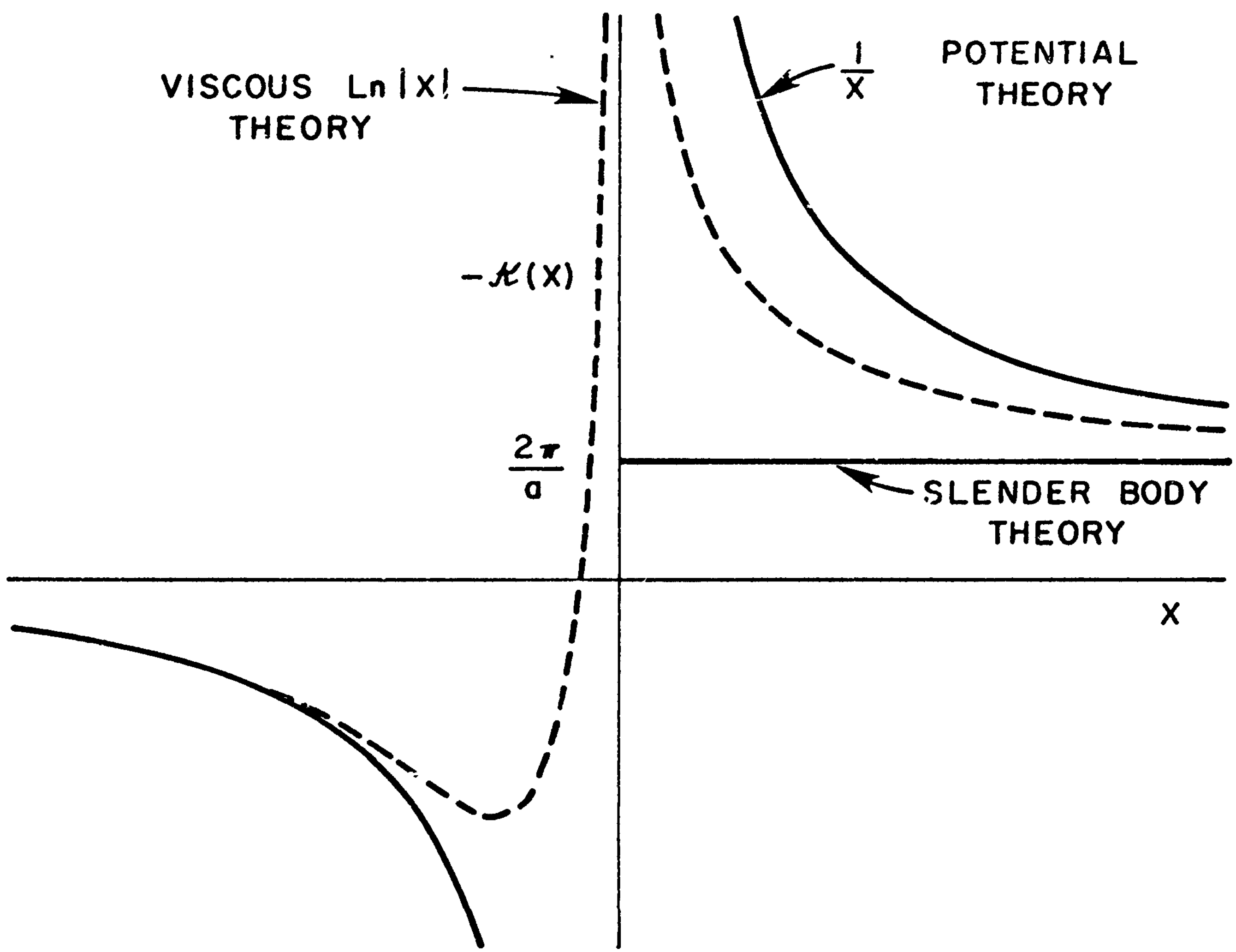

Figure 2.3 Comparison of Inviscid and Viscous Kernel Functions for a Ring-Wing

III. NUMERICAL RESULTS FOR A RING-WING AIRFOIL IN STEADY FLOW

To calculate the load distribution on a ring-wing, it is convenient to introduce dimensionless variables as follows:

$$X = \frac{2x}{\ell} \qquad (3.1)$$

$$\tau = \frac{4a}{\ell} = \frac{\text{diameter}}{\text{half length}} \qquad (3.2)$$

$$k = \tau / \ X^2 + \tau^2 \qquad (3.3)$$

$$\sigma = \frac{V_\infty \ell}{4\nu} \quad \text{Reynolds Number referred to } \ell/4 \qquad (3.4)$$

If we choose the origin of coordinates to be on the axis at the midpoint on the body axis, then the normalized integral equation can be written in the following form:

$$\frac{1}{2\pi} \int_{-1}^{1} \mathscr{L}(Y)\mathscr{K}(X-Y)dY = \alpha(X) \qquad (3.5)$$

where

$$\alpha(X) = -\ z_0' \quad \text{local angle of attack of the body axis} \qquad (3.6)$$

$$\mathscr{K}(x) = \frac{d}{dx}\mathscr{K}_1(x) \qquad (3.7)$$

and

$$\mathscr{K}_1(x) = \frac{\pi x}{\tau} - kK(k) + e^{\sigma x} \quad K_0(\sigma|x|) \tag{3.8}$$

$$+ \frac{8}{\tau^2} \int_0^{|x|} \quad (|x| \ -2s)T(k')ds$$

with

$$T(k') = \frac{1}{k'} \left[\left(1 - \frac{k'^2}{2}\right) K(k') - E(k')\right] \tag{3.9}$$

$$k' = \tau / \sqrt{s^2 + \tau^2} \tag{3.10}$$

We refer to $\mathscr{K}_1(x)$ as the integrated kernel.

The overall normal force and moment coefficients have the following integral representations:

$$C_N = \int_{-1}^{1} \mathscr{L}(X)dX = \frac{\text{Lift}}{\left(\frac{1}{2}\,\rho_\infty v_\infty^2\right)\left(\pi a \ell\right)} \tag{3.11}$$

$$C_M = -\int_{-1}^{1} \quad X\mathscr{L}(X)dX = \frac{\text{Moment}}{\left(\frac{1}{2}\,\rho_\infty v_\infty^2\right)\left(\pi a \ell\right)\left(\frac{\ell}{2}\right)} \tag{3.12}$$

The reference area is chosen to be one half the surface area of the ring-wing. The moment is measured positive nose up about the center of the ring.

The normalized integral equation (3.5) is precisely of the same form as the equation of two-dimensional viscous thin airfoil theory (see Reference 1, Equation (3.25)). The kernel has a logarithmic singularity and this renders a unique solution of Equation (3.5). The load function must admit square root singularities at the fore and aft ends. The equation was reduced to a finite spectral form with an assumed polynomial approximation of the load; i.e.,

$$\mathscr{L} = \sum_{n=0}^{N} A_n \, T_n(X)/\sqrt{1-x^2} \tag{3.13}$$

where T_n is the Chebyshev polynomial of the first kind of degree n (see Reference 6). The numerical algorithm is discussed in detail in Reference 1. The integral equation was solved for a computational Reynolds number of 1000 with 20 Chebyshev polynomials to approximate the load function. It was determined that the solution was adequately converged for this choice of parameters.

Typical calculations of the load distribution are plotted in Figures 3.1, 3.2 and 3.3 for three values of the fineness ratio τ. We note from these calculations that the normal force coefficient is proportional to τ and thus tends to zero as τ tends to zero. This result can also be deduced analytically since the kernel is proportional to $1/\tau$ for very slender bodies. The limiting form is in agreement with the result of slender body theory that predicts a zero lift force on any body of constant radius. For larger values of the fineness ratio (see Figure 3.1 with $\tau = 1$) the load distribution tends to a form that is similar to the lift on a two-dimensional airfoil. This result can also be deduced analytically from the asymptotic form of the kernel (see Equation (2.46)) and is intuitively expected since each section of a thin hoop acts like a two-dimensional airfoil section at a local angle of attack that is a maximum on the upper and lower section and tends to zero on the sides.

To illustrate the last point we note that the normalized ring-wing problem can be expressed in the following form for $\tau \gg 1$:

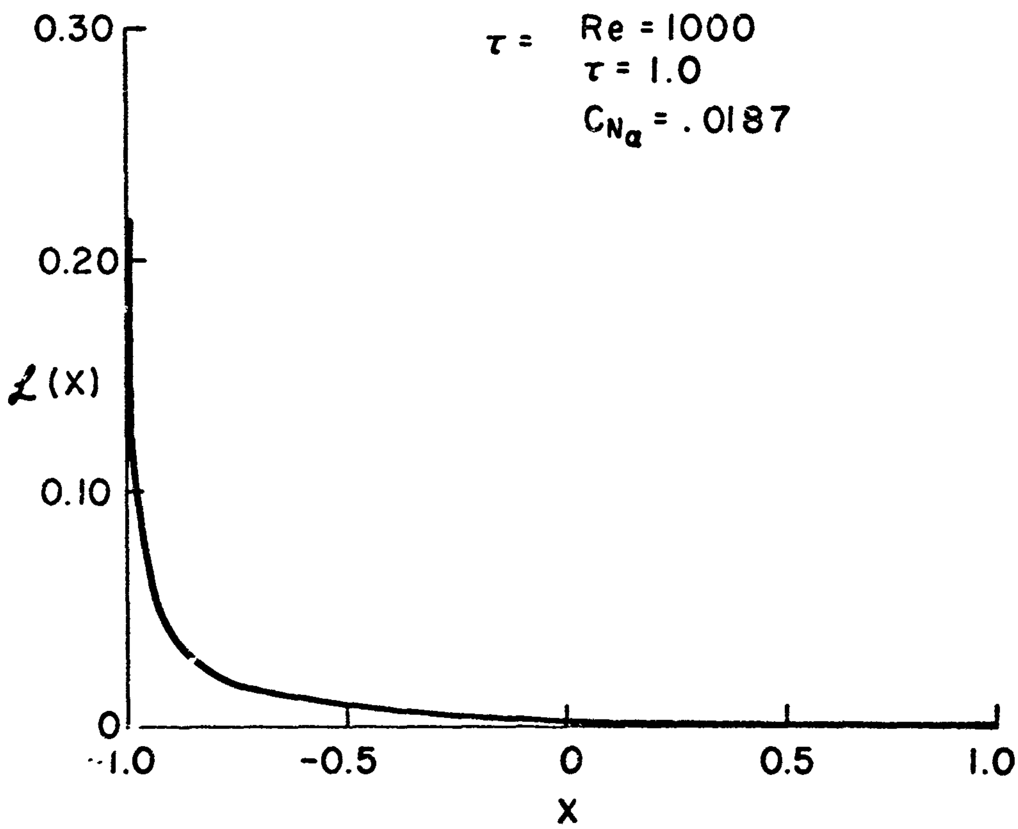

Figure 3.1 Load Distribution on a Ring-Wing at Constant Angle of Attack; τ = 1.0

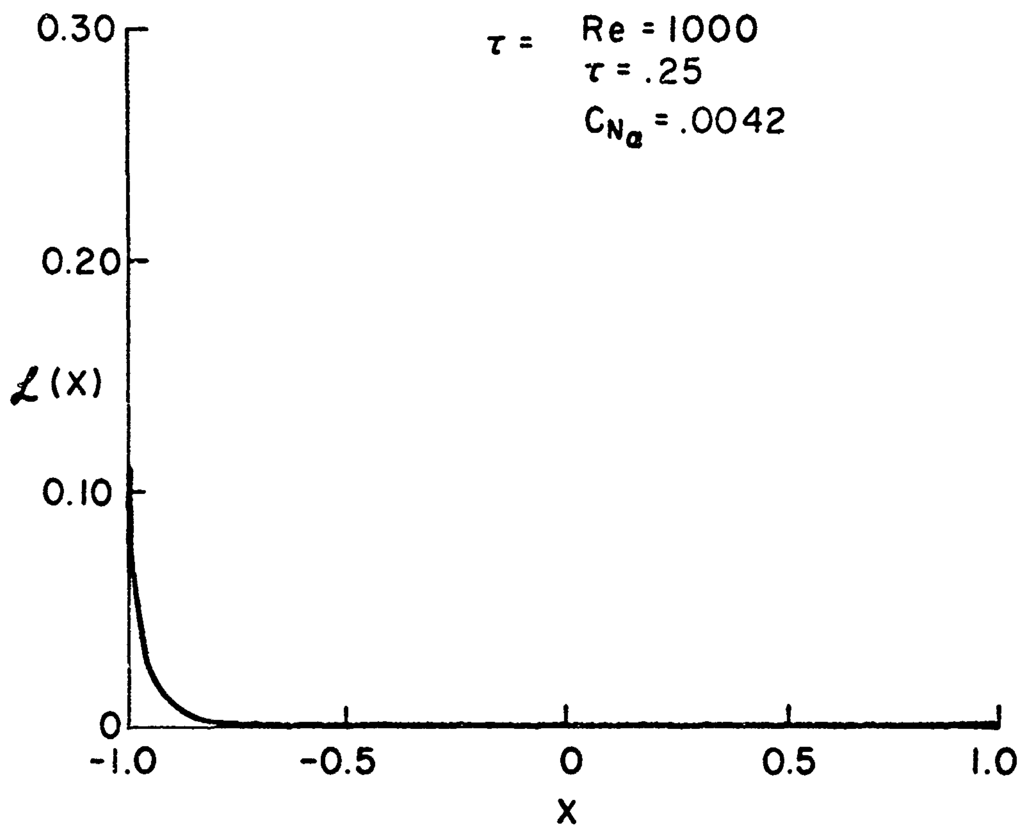

Figure 3.2 Load Distribution on a Ri^g-Wing at Constant Angle of Attack; τ = .25

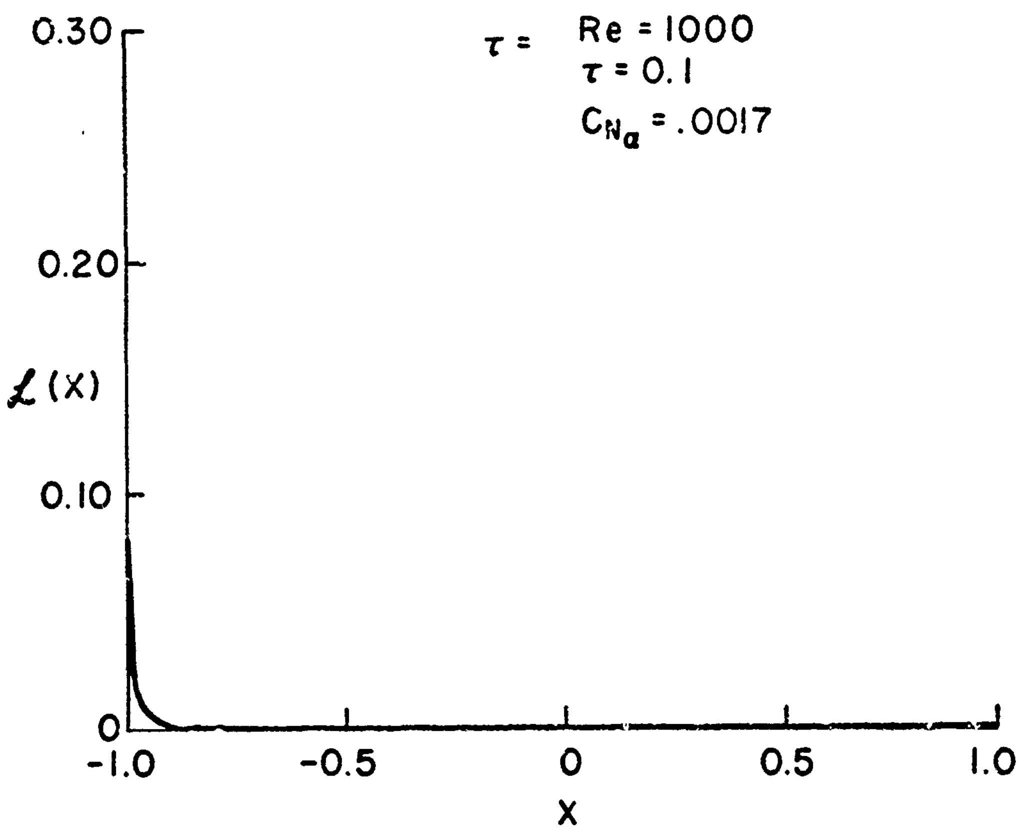

Figure 3.3 Load Distribution on a Ring-Wing at Constant Angle of Attack; τ = .1

$$\frac{1}{2\pi}\int_{-1}^{1}\mathscr{L}(Y)\ \frac{\partial}{\partial X}\left[\ell n|X-Y| + e^{\sigma(X-Y)}\ K_0(\sigma|X-Y|)\right]dY = \alpha - \alpha_i \quad (3.14)$$

where

$$\alpha_i = \frac{1}{2\tau}\int_{-1}^{1}\mathscr{L}(Y)dY = \frac{C_N}{2\tau} \quad (3.15)$$

In the limit $\sigma \to \infty$, Equation (3.13) has the unique solution,

$$\mathscr{L}(x) = \frac{2(1-x)}{\sqrt{1-x^2}}\ (\alpha - \alpha_i) \quad (3.16)$$

and if we integrate the last result over the chord, we obtain

$$C_N = \frac{2\pi\ \alpha}{1+\pi/\tau} \quad (3.17)$$

or

$$C_{N_{\alpha_{\text{Ring-Wing}}}} = \frac{1}{1+\pi/\tau}\cdot C_{N_{\alpha_{\text{2-D Wing}}}} \quad (3.18)$$

The last result is precisely the same form as the lift curve slope of a three-dimensional wing with the typical aspect ratio correction. It also agrees exactly with a result derived by Ribner (Reference 4) with the more elementary lifting line theory.

We compare the last asymptotic result with the experimental data of Flatau (Reference 5) for a 11.7% thick Clark Y airfoil. Typical section data are presented in Figure 3.4 for $\tau \cong 4.9$. From the C_L versus α data we estimate the lift curve slope to be 2.0/rad. From Equation (3.17) the theoretical result is 2.8/rad, a value somewhat larger than the experimental

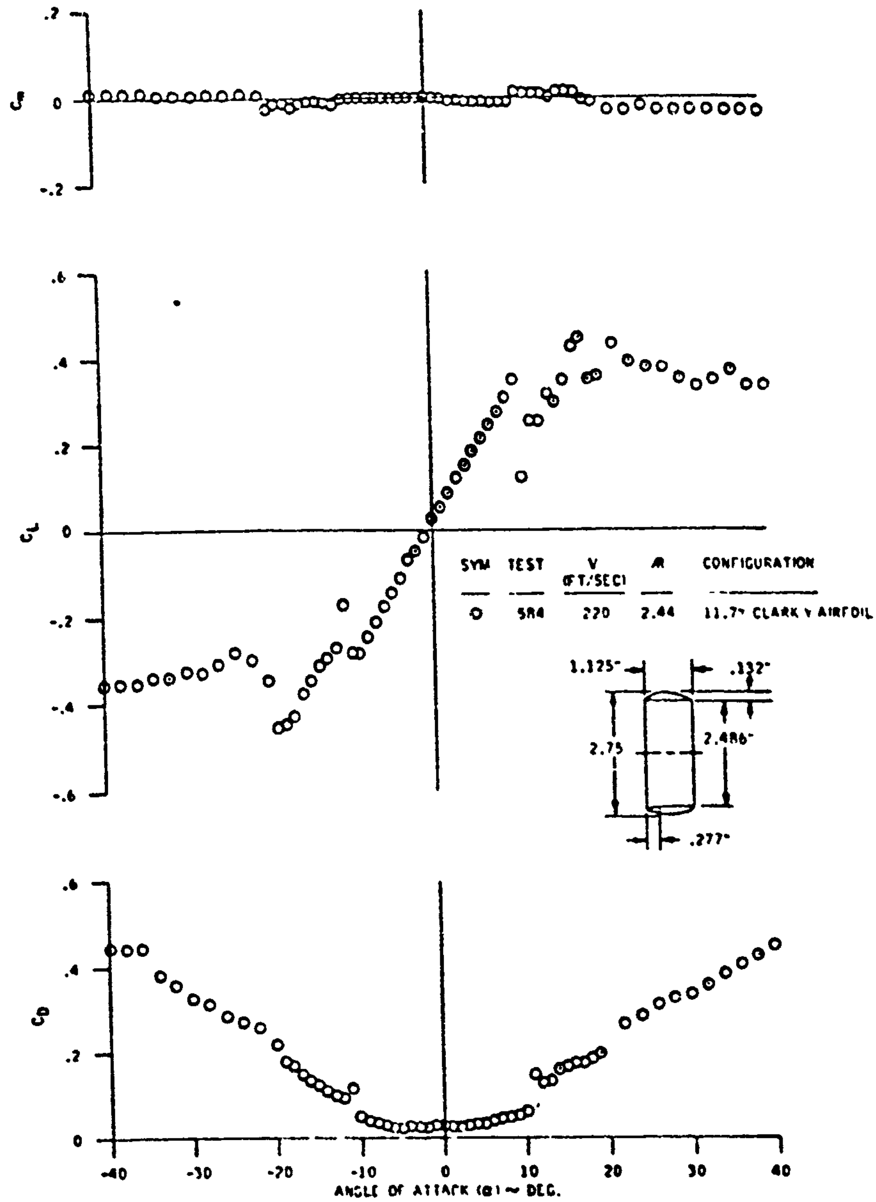

Figure 3.4 Experimental Lift, Moment and Drag Coefficients Versus Angle of Attack for a Ring-Wing with Clark Y Section; $\tau \cong 4.9$ (from Reference 5)

result. The discrepancy is probably due to two sources. First, the theoretical formula is weakly asymptotic in the parameter π/τ. For example, if the next term in the asymptotic expansion of the denominator of Equation (3.17) has an additive term of order $(\pi/\tau)^2$ then for $\tau = 4.9$ (the Flatau value), we can expect a 25% reduction in the theoretical value. Also the two-dimensional section lift coefficient of a Clark Y airfoil is 5.25/rad (Reference 7) compared to the flat plate value of 2π. This represents another 16% reduction in the calculated theoretical value.

IV. SUMMARY AND RECOMMENDATIONS

The viscous theory of lift on bodies of revolution has been derived in detail. The theory is reduced to an integral equation of the airfoil type with a kernel function that has a logarithmic singularity. Detailed calculations of the load distribution and several asymptotic results are presented for a ring-wing airfoil (i.e., the simplest body of revolution). The calculated load distributions are similar to those on a two-dimensional flat plate airfoil for large diameter to length ratios. It is further shown that the asymptotic normal force coefficient on a very large diameter ring-wing is in agreement with a theoretical result of Ribner (Reference 4). For very slender ring-wings the normal force coefficient tends to zero in agreement with the prediction of slender body theory.

It is recommended that the complete steady flow theory for general bodies of revolution be programmed and numerical results obtained for the load distribution. The theoretical calculations should be compared to known experimental data.

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APPENDIX

Evaluation of the Ring-Wing Kernel (Potential Flow)

Consider the following kernel function:

$$\mathscr{K}(x) = \lim_{r\to\rho\to a} \int_{-\infty}^{x} \frac{\partial^2 \mathscr{G}}{\partial r \partial \rho} \, d\xi \tag{A.1}$$

where

$$\mathscr{G} = \int_0^{2\pi} \frac{\cos\theta'}{R'} \, d\theta' \tag{A.2}$$

with

$$R' = \left[\xi^2 + r^2 + \rho^2 - 2r\rho\cos\theta'\right]^{1/2} \tag{A.3}$$

Note that

$$\frac{\partial}{\partial r \partial \rho} \cdot \frac{1}{R} = \frac{\partial^2}{\partial r \partial \xi}\left[-\frac{\xi}{R'} \cdot \frac{\rho - r\cos\theta'}{r^2 + \rho^2 - 2r\rho\cos\theta'}\right] \tag{A.4}$$

Introduce the change of variable

$$\theta' = \pi - \theta \tag{A.5}$$

and substitute Equation (A.4) into Equation (A.1). The final result is

$$\mathscr{K}(x) = \lim_{r\to a} \frac{\partial\phi}{\partial r} \tag{A.6}$$

with

$$\phi = \int_{-\pi}^{\pi} \cos\theta \, d\theta \left(1 + \frac{x}{R}\right)\left(\frac{a+r\cos\theta}{\lambda^2}\right) \tag{A.7}$$

$$R = (x^2 + \lambda^2)^{1/2} \tag{A.8}$$

and

$$\lambda^2 = r^2 + a^2 + 2ra\cos\theta \tag{A.9}$$

To evaluate ϕ, we introduce the variable

$$z = \tan\theta/2 \ , \quad -\infty < z < \infty \tag{A.10}$$

so that

$$\cos\theta = \frac{1-z^2}{1+z^2} \ , \quad d\phi = \frac{2dz}{1+z^2} \tag{A.11}$$

It is readily shown that

$$\phi = \phi_0 + \phi_1 \tag{A.12}$$

where

$$\phi_0 = \frac{2A}{(r-a)} \int_{-\infty}^{\infty} \frac{(1-z^2)}{(1+z^2)^2} \cdot \left(\frac{A-z^2}{A^2+z^2}\right) dz \tag{A.13}$$

$$\phi_1 = g(x)A\int_{-\infty}^{\infty} \frac{(1-z^2)(A-z^2)dz}{(z^2+1)^{3/2}(z^2+B^2)^{1/2}(A^2+z^2)} \tag{A.14}$$

with

$$g(x) = \frac{2x}{(r-a)[x^2+(r-a)^2]^{1/2}} \tag{A.15}$$

$$A = \frac{r+a}{r-a} \tag{A.16}$$

and

$$B = \left[\frac{x^2 + (r+a)^2}{x^2 + (r-a)^2}\right]^{1/2} \tag{A.17}$$

Now

$$\frac{(1-z^2)(A-z^2)}{(1+z^2)(z^2+A^2)} = 1 + \frac{2}{A-1} \cdot \frac{1}{z^2+1} - \frac{A(A^2+1)}{A-1} \cdot \frac{1}{z^2+A^2} \tag{A.18}$$

so that

$$\phi_o = \frac{2}{(r-a)}\int_{-\infty}^{\infty} dz\left\{\frac{1}{1+z^2} + \frac{2}{(A-1)} \cdot \frac{1}{(1+z^2)^2}\right. \tag{A.19}$$

$$\left. - \frac{A(A^2+1)}{A-1} \cdot \frac{1}{(1+z^2)(z^2+A)}\right\}$$

All of the integrals in Equation (A.19) may be found in Reference 8, pp. 13 through 17. The final result is

$$\phi_0 = \frac{2\pi}{(r-a)(A+1)} = \frac{\pi}{r} \tag{A.20}$$

Thus

$$\lim_{r\to a} \frac{\partial\phi_0}{\partial r} = -\frac{\pi}{a^2} \tag{A.21}$$

To evaluate ϕ_1, we first note that

$$\frac{1}{r-a} = \frac{A-1}{2a} \tag{A.22}$$

Thus

$$\phi_1 = \frac{x}{a[x^2 + (r-a)^2]^{1/2}} \; G(A,B) \tag{A.23}$$

where

$$G(A,B) = (A-1)\int_{-\infty}^{\infty} \frac{(1-z^2)(A-z^2)}{(z^2+1)(z^2+A^2)} \frac{dz}{(z^2+B^2)^{1/2}(z^2+1)^{1/2}} \tag{A.24}$$

or after using Equation (A.18) we can write

$$G(A,B) = (A-1)G_0 + 2G_1 - A(A^2+1)G_2 \tag{A.25}$$

where

$$G_0 = \int_{-\infty}^{\infty} \frac{dz}{(z^2+1)^{1/2}(z^2+B^2)^{1/2}} \tag{A.26}$$

$$G_1 = \int_{-\infty}^{\infty} \frac{dz}{(z^2+1)^{3/2}(z^2+B^2)^{1/2}} \tag{A.27}$$

$$G_2 = \int_{-\infty}^{\infty} \frac{dz}{(z^2+A^2)(z^2+1)^{1/2}(z^2+B^2)^{1/2}} \tag{A.28}$$

Consider the integral

$$F = \int_{-\infty}^{\infty} \frac{dz}{(z^2+t^2)^{1/2}(z^2+B^2)^{1/2}} \quad , \quad B > t \tag{A.29}$$

From Reference 8, p. 51, we have

$$F = \frac{2}{B} \, K\left(\frac{\sqrt{B^2-t^2}}{B}\right) \tag{A.30}$$

Thus

$$G_0 = \lim_{t \to 1} F = \frac{2}{B} \, K(k) \tag{A.31}$$

with

$$k = \frac{\sqrt{B^2-1}}{B} \tag{A.32}$$

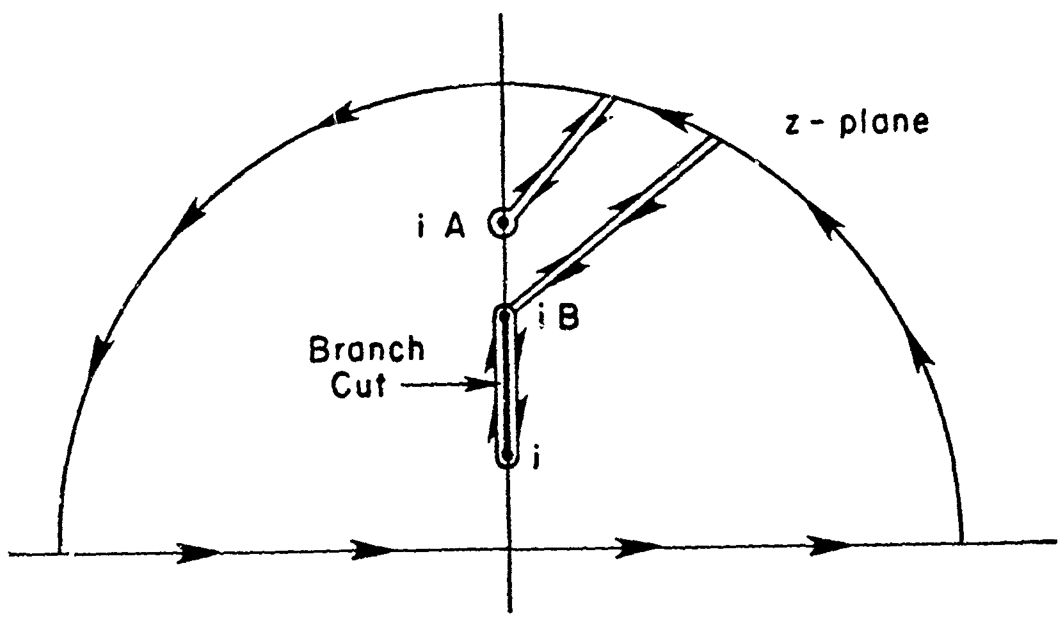
and

$$G_1 = \lim_{t \to 1} \left(- \frac{\partial F}{\partial t} \right) = \frac{2}{B^3} K'(k) \tag{A.33}$$

where K(k) in the standard elliptic integral of the first kind; i.e.,

$$K(k) = \int_0^{\pi/2} \frac{d\phi}{\sqrt{1-k^2 \sin^2\phi}} \tag{A.34}$$

In the complex z plane the integrand of G_2 has branch points at $z = \pm i$ and $z = \pm iB$ and simple poles at $z = \pm iA$. With the contour shown in the sketch we can replace G_2 by an integral around the branch cut and the pole at



$z = iA$. The result is

$$G_2 = \frac{-\pi}{A(A^2-1)^{1/2}(A^2-B^2)^{1/2}} + 2 \int_1^B \frac{dt}{(A^2-t^2)(t^2-1)^{1/2}(B^2-t^2)^{1/2}} \tag{A.35}$$

where the first term is the residue at $z = iA$. The integral is given in Reference 8, p. 41, so that

$$G_2 = -\frac{\pi}{A(A^2-1)^{1/2}(A^2-B^2)^{1/2}} + \frac{2}{BA^2}K(k) + \frac{2}{BA^2(A^2-1)}\,\Pi\left(-\frac{A^2k^2}{A^2-1}, k\right) \quad (A.36)$$

where Π is the elliptic integral of the third kind; i.e.,

$$\Pi(\rho,k) = \int_0^1 \frac{dt}{(1+\rho t^2)\sqrt{(1-t^2)(1-k^2t^2)}}, \quad 0 < k < 1 \quad \rho > -1 \quad (A.37)$$

With Equations (A.31), (A.33) and (A.37) we can write G in the following form:

$$G(A,B) = -\frac{2}{B}\left(1+\frac{1}{A}\right)K(k) + \frac{4}{B^3k}K'(k) \quad (A.38)$$

$$-\frac{2}{AB}\left(\frac{A^2+1}{A^2-1}\right)\Pi\left(-\frac{A^2k^2}{A^2-1}, k\right) + \pi\frac{A^2+1}{(A^2-1)^{1/2}(A^2-B^2)^{1/2}}$$

The final step in the derivation of the ring-wing kernel is to differentiate Equation (A.38) with respect to r and take the limit as $r \to a$. It is useful to note that

$$\lim_{r\to a} \frac{\partial}{\partial r}\frac{1}{A} = \frac{1}{2a} \quad (A.39)$$

and

$$\lim_{r\to a} \frac{\partial}{\partial r}\frac{1}{A^2} = 0 \quad (A.40)$$

Also

$$\lim_{r\to a} \Pi\left(-\frac{A^2k^2}{A^2-1}, k\right) = \Pi\,(-k^2,k) = \frac{E(k)}{1-k^2} \qquad (A.41)$$

and

$$\left.\frac{\partial k}{\partial r}\right|_{r=a} = x^2/(x^2+4a^2)^{3/2} \qquad (A.42)$$

where E(k) is the elliptic integral of the second kind; i.e.,

$$E(k) = \int_0^{\pi/2} \sqrt{1-k^2\sin^2\phi}\ d\phi \qquad (A.43)$$

After a considerable amount of detailed algebraic manipulation, the complete ring-wing kernel function can be expressed in the following form:

$$\mathscr{K}(x) = \frac{1}{a^2}\left\{-\pi - \left[\frac{E(k)}{\sqrt{1-k^2}}\left(\frac{2}{k} - k\right)^2 - \frac{4}{k^2}(1-k^2)^{3/2} K(k)\right] \operatorname{sgn} x\right\} \qquad (A.44)$$

where

$$k = \frac{2a}{\sqrt{x^2+4a^2}}, \quad 0 < k < 1 \qquad (A.45)$$

We conclude this appendix by stating two asymptotic properties of the kernel. First, we note that

$$|x| \gg 2a \longrightarrow k = \frac{2a}{|x|} + 0\left(\frac{a^3}{|x|^3}\right) \qquad (A.46)$$

and the kernel has the simple approximation

$$K(x) = -\frac{2\pi}{a^2} H(x) \quad \text{for} \quad |x| \gg 2a \qquad (A.47)$$

where H(x) is the heaviside step function. We refer to this asymptotic form as the slender body kernel. The second result is obtained for $x \ll 2a$ or $k \to 1$. In this case we identify the Cauchy singularity in the kernel; i.e.,

$$K(x) \cong -\frac{2}{ax} \quad \text{for } x \ll 2a \qquad (A.48)$$

Further discussion of the kernel may be found in the main text.